Boundary Damping Applied to Vibrating Continuous Systems

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Abstract. In this paper initial-boundary value problems for a linear, and a weakly nonlinear string (or wave) equation are considered. One end of the string is assumed to be fixed and the other end of the string is attached to a spring-mass-dashpot system, where the damping generated by the dashpot is assumed to be small. This problem can be regarded as a simple model describing oscillations of flexible structures such as overhead power transmission lines. For a linear problem a semigroup approach will be used to show the well-posedness of the problem as well as the asymptotic validity of formal approximations of the solution on long time-scales. It is also shown how a multiple time-scales perturbation method as described in [1] can be used effectively to construct asymptotic approximations of the solution on long timescales.

Introduction

There are examples of flexible structures such as suspension bridges, overhead transmission lines, dynamically loaded helical springs that are subjected to oscillations due to different causes, such as windflows or earthquakes. In some cases, the so-resulted oscillations may cause undesirable behaviour. For instance, in 1940 the Tacoma Narrows Bridge collapsed completely because an 18.9 m/s wind - flow induced an 0.23 Hz torsional oscillation of the bridge deck. More examples of undesirable oscillations are the oscillations of the stays of the Erasmus Bridge in Rotterdam during stormy and rainy weather. Simple models which describe these oscillations can be expressed in initial-boundary value problems for wave equations like in Keller and Kogelman [2], van Horssen [3] or for beam equations like in Castro and Zuazua [4], and Boertjens and van Horssen [5]. To suppress the oscillations various types of boundary damping can be applied such as described in Castro and Zuazua [4], Morgul et al. [6].

In some flexible structures (such as an overhead transmission line or a cable of a suspension bridge) various types of wind-induced mechanical vibrations can occur. Vortex shedding for instance causes usually high frequency oscillations with small amplitudes, whereas low frequency vibrations with large amplitudes can be caused by flow-induced oscillations (galloping) of cables on which ice or snow has accreted. These vibrations can give rise to material fatigue. To suppress these oscillations various types of dampers have been applied in practice (see for instance Wang et al. [7].

In most cases simple, classical boundary conditions are applied such as in Boertjens and van Horssen [5], Keller and Kogelman [2], van Horssen [3] to construct approximations of the oscillations. For more complicated, non-classical boundary conditions, see for instance in Castro and Zuazua [4], Morgul [6], it is usually not possible to construct explicit approximations of the oscillations. In this paper we will study such an initial-boundary value problem with a non-classical boundary condition and we will construct explicit asymptotic approximations of the solution, which are valid on a long time-scale. The main problem of this paper is to study how efficiently these boundary dampers work. The method which can be used to investigate these problems are multiple timescales methods (as used in Boertjens and van Horssen [5], Keller and Kogelman [2], van Horssen [3]), Galerkin truncation methods, and combinations of these methods. From the asymptotic point of view it is also interesting to study the convergence properties of the applied perturbation methods for these types of initial - boundary value problems. We will consider a string which is fixed at $x=0$ and attached to a spring-mass-dashpot system at $x=\pi$.

To derive a model for flexible structures such as suspension bridges or overhead transmission lines we refer to van Horssen [3] and Boertjens and van Horssen [5]. It is assumed that l (the length of the string), ρ (the mass-density of the string), T (the tension in the string), $\tilde{m}$ (the mass in the spring-mass-dashpot system), $\tilde{\gamma}$ (the stiffness of the spring), and $\tilde{\alpha}$, β (the damping coefficients of the dashpot), p^2 (for instance, the stiffness of the stays of the bridge) are all positive constants. Furthermore, we only consider the vertical displacement $\tilde{u}(x,\tilde{t})$ of the string, where x is the place along the string, and $\tilde{t}$ is time. After applying a simple rescaling in time and in displacement $\tilde{t}=\sqrt{T/\rho}t$, $\tilde{u}(x,\tilde{t})=u(x,t)$; putting $\tilde{m}=\rho m$, $\tilde{\gamma}=\gamma.T$, and $\tilde{\alpha}=\sqrt{T\rho}\alpha$ we obtain as a simple model for the oscillations of the string the following initial-boundary value problem

$$\begin{aligned} u_{tt}-u_{xx}+p^2u &= f(x,u,u_t), \\ u(0,t)=0,\ u_x &= -(mu_{tt}+\gamma u \\ &\quad +\alpha u_t),\ x=l,\ t\geq 0, \\ u(x,0)=\phi(x), u_t(x,0) &= \psi(x) \end{aligned}$$

where ε is a small parameter with $0 < \varepsilon << 1$ and where the function f is an external force (for instance a wind force). The functions ϕ and ψ represent the initial displacement of the string and the initial velocity of the string respectively. Different cases are considered for f, m, γ, α and β. In this paper we will consider the following four cases, namely;

1. $p^2 = 0, \beta = 0, f(x,u,u_t) \equiv 0, m, \gamma = O(1),$
   $\alpha = O(\varepsilon).$
2. $p^2 \neq 0, \beta = 0, f(x,u,u_t) = u_t - \frac{1}{3}u_t^3,$
   $m, \gamma, \alpha = O(\varepsilon).$
3. $p^2 \neq 0, f(x,u,u_t) \equiv 0, m, \gamma = 0, \alpha, \beta = O(\varepsilon).$
4. $p^2 = 0, f(x,u,u_t) = u_t - \frac{1}{3}u_t^3, m, \gamma, \beta = 0,$
   $\alpha = O(\varepsilon).$

For the first case and third case a semigroup approach, as described in Goldstein [8], can be used to show the well-posedness of the problem for suitable initial conditions as well as to prove the asymptotic validity of formal approximations of the solution on long time-scales. To construct an approximation of the solution of the problem a perturbation technique will be used. The first, second, and fourth case are regularly perturbed problems, whereas the third case is a singularly perturbed problem. In this paper we will present the results as obtained so far. For details we refer to our papers [9,10,11].

Methods.

In the applied perturbation scheme it is assumed that the solution of the problem can be expanded in a power series in ε. If a naive expansion is used, that is, if it is assumed that the solution can be written as

$$u(x,t;\varepsilon) = u_0(x,t) + \varepsilon u_1(x,t) + \cdots,$$

it may turn out that u_0, u_1, u_2, and so on, may contain terms growing in $x/\varepsilon, t/\varepsilon, t, \varepsilon t$, or $\varepsilon^2 t$. Of course, the approximation is still valid for very small values of *t* and *x*. But it is not valid anymore for large values of *t* and *x*. These terms are the so-called secular terms. To avoid the error caused by these secular terms it is convenient to scale the time variable *t* and the space variable *x* by introducing new variables $\bar{x} = x/\varepsilon$, $\bar{t} = t/\varepsilon$, $\tau = \varepsilon t$, $\mu = \varepsilon^2 t$, and so on. To remove secular terms occurring in u_0, u_1, u_2 and so on it is assumed that the approximation of u is a function of $\bar{x}, x, \bar{t}, t, \tau$ and μ, and so on. Then *u* is expanded in a power series in ε, that is,

$$u(x,t;\varepsilon) = v_0(\bar{x},x,\bar{t},t,\tau,\mu) + \varepsilon v_1(\bar{x},x,\bar{t},t,\tau,\mu) + \varepsilon^2 v_2(\bar{x},x,\bar{t},t,\tau,\mu)\cdots.$$

This method is called the method of multiple scales.

Results.

For the first case, using a semigroup approach it can be proved that the problem is wellposed for $0 < x < 1$ and for $t \geq 0$. Although the problem is linear the construction of these approximations is far from being elementary because of the complicated, non-classical boundary condition. Using some kind of balancing procedure we solve the linear wave equation and construct approximations. In fact, the procedure is an extension of the classical way to solve a linear wave equation using the method of separation of variables. We observe that although the problem we consider is homogeneous, the technique used to approximate the solution can be applied to nonhomogeneous problems as well. To construct a formal approximation the scallings which are needed are x, t, and τ. It has been shown that this type of boundary damper makes the zero solution stable but not uniform. It has also been shown that the approximation is an asymptotic one on a time - scale of order ε^{-1}.

For the second case, to construct a formal asymptotic approximations of the exact solution we use a two-timescales perturbation method, that is, it is assumed that the approximation is a function of x, t and $\tau = \varepsilon t$. The formal asymptotic approximation is expressed in the form of an infinite series. It has been shown that for all values of $p^2 > 0$ mode-interactions only occur between modes with non-zero initial energy up to $O(\varepsilon)$. This implies that truncation is allowed to those modes that have non-zero initial energy up to $O(\varepsilon)$. For the damping parameter $\alpha \geq \pi/2$ it has been shown that all solutions tend (up to $O(\varepsilon)$) to zero as $t \to \infty$. For $0 < \alpha < \pi/2$ it can be shown that the string system usually will oscillate in only one mode (up to $O(\varepsilon)$) as $t \to \infty$. This indicates that the applied damper at the boundary is an efficient one.

For the third case, the presence of the term u_{xt} in the boundary condition at $x = \pi$ will give rise to a singularly perturbed problem. In fact a characteristic layer near $x = \pi$ will play an important role in the construction of an approximation of the exact solution. To construct formal approximations of the solution the method of multiple scales will be used. It is clear from the boundary condition at $x = \pi$ that the tangent of the initial displacement of string near $x = \pi$ is of order $O(\varepsilon)$. It is, however, not clear what scalings are to be needed to approximate the solution of the problem. But for $\alpha = 0$ the exact solution can be determined. We observe that for $\alpha = 0$ the solution consists of two parts. The first part of the solution only plays a significant role (in the (x,t)-plane) in an order ε neighborhood of $x = \pi$. The second part of the solution in fact describes the vibrations of a string with a Dirichlet boundary condition at $x = 0$ and a Neumann boundary condition at $x = \pi$. It is clear from this solution that the scallings we need to construct an approximation solution are: $x, \bar{x} = x/\varepsilon, t$, and $\bar{t} = t/\varepsilon$. Based upon the results obtained from the first case, the case 2, and the case 3 of this paper it is most likely that for $\alpha \neq 0$ we need additional scalings $\tau = \varepsilon t, \mu = \varepsilon^2 t, \cdots$. It follows from the exact solution of the initial-boundary value problem with $\alpha = 0$ that in order to describe the characteristic layer correctly we have to construct an approximation at least up to order ε^2. For that reason for this case a secular free approximation will be constructed up to order ε^2, that is, $\bar{u} = u_0 + \varepsilon u_1 + \varepsilon^2 u_2$. It has been shown that the zero solution is uniformly stable. So, it will be effective to use such a boundary damper to damp the oscillations of the string.

For the last case, the problem can be regarded as a simple model of the galloping oscillations of overhead power transmission lines in a windfield. For this problem the truncation method can not be applied. Instead of the Fourier-series method we can use the method of characteristic coordinates (in combination with a multiple-timescales perturbation method). Again it can be shown that for $\alpha \geq \pi/2$ all solutions will tend to zero (up to $O(\varepsilon)$) as $t \to \infty$. For $0 < \alpha < \pi/2$ it can be shown that the solution will tend to a standing triangular wave as $t \to \infty$ (with a vanishing amplitude as α tends to $\pi/2$. The calculations for this problem are more complicated then the ones presented for the second case.

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