On the weakly damped vibrations of a string attached to a spring-mass-dashpot system

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Abstract: In this paper an initial-boundary value problem for a homogeneous string (or wave) equation is considered. One end of the string is assumed to be fixed and the other end of the string is attached to a spring-mass-dashpot system, where the damping generated by the dashpot is assumed to be small. This problem can be regarded as a simple model describing oscillations of flexible structures such as overhead power transmission lines. A semigroup approach will be used to show the well-posedness of the problem as well as the asymptotic validity of formal approximations of the solution on long time-scales. A multiple time-scales perturbation method will be used to construct asymptotic approximations of the solution. Although the problem is linear the construction of these approximations is far from being elementary because of the complicated, non-classical boundary condition.

Key Words: boundary damping,semigroup, asymptotics,two-timescales perturbation method.

1. INTRODUCTION

There are examples of flexible structures such as suspension bridges, overhead transmission lines, dynamically loaded helical springs that are subjected to oscillations due to different causes. Simple models which describe these oscillations can be expressed in initial-boundary value problems for wave equations (like in Krol,1989; Durant,1960; van Horssen,1988; Cox and Zuazua,1995; Rao,1993) or for beam equations (like in Boertjens and van Horssen, 2000; Conrad and Morgül,1998; Rao,1995).

In most cases simple, classical boundary conditions are applied (such as in Boertjens and van Horssen,2000; van Horssen,1988) to construct approximations of the oscillations. For more complicated, non-classical boundary conditions (see for instance Durant,1960; Morgül et al.,1994; Castro and Zuazua,1998; Cox and Zuazua,1995; Rao,1993; Rao,1995) it is usually not possible to construct explicit approximations of the oscillations. In this paper we will study such an initial-boundary value problem with a non-classical boundary condition and we will construct explicit asymptotic approximations of the solution, which are valid on a long time-scale. We will consider a string which is fixed at $x = 0$ and attached to a spring-mass-dashpot system at $x = 1$ (see also figure).

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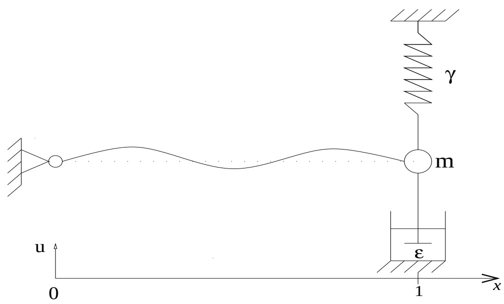

Fig. A simple model of a string fixed at $x = 0$ and attached to a spring-mass-dashpot system at $x = 1$.

It is assumed that ρ (the mass-density of the string), T (the tension in the string), $\tilde{m}$ (the mass in the spring-mass-dashpot system), $\tilde{\gamma}$ (the stiffness of the spring), and $\tilde{\epsilon}$ (the damping coefficient of the dashpot with $0 < \tilde{\epsilon} \ll 1$) are all positive constants. Furthermore, we only consider the vertical displacement $\tilde{u}(x, \tilde{t})$ of the string, where x is the place along the string, and $\tilde{t}$ is time. Gravity and other external forces are neglected.

After applying a simple rescaling in time and in displacement ($\tilde{t} = \sqrt{\frac{T}{\rho}}t$, $\tilde{u}(x, \tilde{t}) = u(x, t)$; putting $\tilde{m} = \rho.m$, $\tilde{\gamma} = \gamma.T$, and $\tilde{\epsilon} = \sqrt{T\rho}\epsilon$) we obtain as a simple model for the oscillations of the string the following initial-boundary value problem

$$\begin{aligned} u_{tt} - u_{xx} &= 0, \;\; 0 < x < 1, \; t > 0, && (1)\\ u(0,t) &= 0, \; t \geq 0, && (2)\\ mu_{tt}(1,t) + \gamma u(1,t) + u_x(1,t) &= -\epsilon u_t(1,t), \; t \geq 0, && (3)\\ u(x,0) &= \phi(x), \; 0 < x < 1, && (4)\\ u_t(x,0) &= \psi(x), \; 0 < x < 1, && (5) \end{aligned}$$

where m and γ are positive constants, and where ϵ is a small parameter with $0 < \epsilon \ll 1$. The functions ϕ and ψ represent the initial displacement of the string and the initial velocity of the string respectively. The term $u_x(1, t)$ in boundary condition (3) represents the force by the string acting on the mass.

In this paper we will prove the well-posedness of the initial-boundary value problem (1) - (5), and we will construct explicit, asymptotic approximations of the solution up to order ϵ on a time-scale of order ϵ^{-1} . This paper is organized as follows. In section 2 we first study the undamped initial - boundary value problem (1) - (5) with $\epsilon = 0$. In section 3 of this paper a boundedness property of the solution is discussed. By using a semigroup approach we show in section 4 that for $0 < \epsilon \ll 1$ the problem (1) - (5) is well-posed for all $t \geq 0$. In section 5 a formal approximation of the solution of (1) - (5) is constructed using a multiple timescales perturbation method. The asymptotic validity of this formal approximation will be proved in section 6 on a time-scale of order ϵ^{-1}. Finally in section 7 some remarks will be made and some conclusions will be drawn.

2. THE UNDAMPED PROBLEM (1) - (5) WITH $\epsilon = 0$

In this section the method of separation of variables will be used to solve problem (1) - (5) with $\epsilon = 0$, that is,

$$\begin{aligned} u_{tt} - u_{xx} &= 0, \ \ 0 < x < 1, \ t > 0, && (6)\\ u(0,t) &= 0, \ t \geq 0, && (7)\\ mu_{tt}(1,t) + \gamma u(1,t) + u_x(1,t) &= 0, \ t \geq 0, && (8)\\ u(x,0) &= \phi(x), \ 0 < x < 1, && (9)\\ u_t(x,0) &= \psi(x), \ 0 < x < 1. && (10) \end{aligned}$$

The solution of this problem will play an important role in section 5. We now look for a nontrivial solution of the PDE (6) and the BCs (7) - (8) in the form $X(x)T(t)$. By substituting this form into (6) - (8) we obtain a boundary value problem for $X(x)$:

$$\begin{aligned} -X''(x) &= \lambda X(x), && (11)\\ X(0) &= 0, && (12)\\ X'(1) &= (m\lambda - \gamma)X(1), && (13) \end{aligned}$$

and the following problem for $T(t)$:

$$-T''(t) = \lambda T(t). \tag{14}$$

It can be shown elementarily that the eigenvalue problem (11) - (13) has infinitely many, isolated, real-valued, and positive eigenvalues. We will omit the elementary proofs and we refer the reader to (Strauss,1992) for similar proofs. The eigenvalues are the roots of

$$\cot(\sqrt{\lambda}) = \frac{m\lambda - \gamma}{\sqrt{\lambda}}. \tag{15}$$

We denote these roots by λ_n, and it can be deduced from (15) that $(n-1)\pi < \sqrt{\lambda_n} < n\pi$, $n = 1, 2, 3, \cdots$. From (11) - (13) and (15) the eigenfunctions $X_n(x)$ can now be determined, yielding $X_n(x) = A_n \sin(\sqrt{\lambda_n}x)$, where A_n is a constant. It can also be shown elementarily that two different eigenfunctions belonging to two different eigenvalues are orthogonal with respect to the inner product as defined by

$$< W_1(x), W_2(x) >= \int_0^1 \Big[1 + m\delta(x-1)\Big] W_1(x)W_2(x)dx, \tag{16}$$

where $\delta(x-1) = 0$ for $x \neq 1$ and $\int_0^1 \delta(x-1)dx = 1$. From (14) $T_n(t)$ can now be determined for each λ_n, and so inifinitely many nontrivial solutions of the PDE(6) and BCs (7) - (8) have been determined in the form $X_n(x)T_n(t)$. Using the superposition principle, the inner product (16), and the initial values (9) - (10) we finally obtain the solution of the undamped problem (6) - (10), yielding

$$u(x,t) = \sum_{n=1}^{\infty} \Big(A_n \sin(\sqrt{\lambda_n}t) + B_n \cos(\sqrt{\lambda_n}t) \Big) \sin(\sqrt{\lambda_n}x), \tag{17}$$

where

$$A_n = \frac{\int_0^1 [1 + m\delta(x-1)]\psi(x) \sin(\sqrt{\lambda_n}x)dx}{\int_0^1 [1 + m\delta(x-1)] \sin^2(\sqrt{\lambda_n}x)dx}, \tag{18}$$

and

$$B_n = \frac{1}{\sqrt{\lambda_n}} \frac{\int_0^1 [1 + m\delta(x-1)]\phi(x) \sin(\sqrt{\lambda_n}x)dx}{\int_0^1 [1 + m\delta(x-1)] \sin^2(\sqrt{\lambda_n}x)dx}. \tag{19}$$

3. THE ENERGY AND THE BOUNDEDNESS OF SOLUTIONS.

The energy of the string with the mass-spring system is defined to be

$$E(t) = \int_0^1 \left(\frac{1}{2}u_t^2(x,t) + \frac{1}{2}u_x^2(x,t)\right)dx + \frac{1}{2}mu_t^2(1,t) + \frac{1}{2}\gamma u^2(1,t). \tag{20}$$

It should be observed that the energy consists of the kinetic and the potential energies of the string and the mass-spring system. Elementarily it can be shown that $\frac{dE}{dt} = -\epsilon u_t^2(1,t) \leq 0$. So, $E(t) \leq E(0)$ for all $t \geq 0$. By using the Cauchy-Schwarz inequality it then follows that

$$|u(x,t)| \quad = \quad \left|\int_0^x u_s(s,t)ds\right| \leq \sqrt{\int_0^1 u_s^2(s,t)ds} \leq \sqrt{2E(t)} \leq \sqrt{2E(0)}.$$

And so, $u(x,t)$ is bounded if the initial energy is bounded.

4. WELL-POSEDNESS OF THE PROBLEM (1) - (5)

In this section we will show that the initial - boundary value problem (1) - (5) with $0 < \epsilon \ll 1$ is well-posed for all $t > 0$. To show the well - posedness we will use a semigroup approach. For that reason we introduce the following auxiliary functions defined as follows:

$$a(t) = u(\bullet,\, t), \quad b(t) = u_t(\bullet,\, t), \text{ and } \quad \eta(t) = mu_t(1,t). \tag{21}$$

For simplicity, we denote a, b, η for $a(t)$, $b(t)$, $\eta(t)$, respectively. Differentiating these functions with respect to t we obtain

$$\begin{pmatrix} a_t \\ b_t \\ \eta_t \end{pmatrix} = \begin{pmatrix} b \\ a_{xx} \\ -\left(\gamma a(1) + a_x(1) + \frac{\epsilon}{m}\eta\right) \end{pmatrix}. \tag{22}$$

Next, we also define some function spaces, i.e: $\mathcal{V} := \{a \in H^1[0,1],\, a(0) = 0\}$, and $\mathcal{H} := \{y(t) = (a,b,\eta) \in \mathcal{V}\times L^2[0,1] \times \Re\}$. Now we equip the space $\mathcal{H}$ with the inner product $\langle\cdot\,,\,\cdot\rangle : \mathcal{H}\times\mathcal{H} \to \Re$ defined by

$$\langle y\,,\,\tilde{y}\rangle := \int_0^1 (a_x\tilde{a}_x + b\tilde{b})dx + \gamma a(1)\tilde{a}(1) + \frac{1}{m}\eta\tilde{\eta}, \tag{23}$$

where $y = (a,b,\eta)$ and $\tilde{y} = (\tilde{a},\tilde{b},\tilde{\eta})$ are in $\mathcal{H}$. Observe that this inner product is based upon the energy of the string (see also (20)). For that reason we call the space the energy space $\mathcal{H}$. The energy space $\mathcal{H}$ together with the inner product $\langle\cdot\,,\,\cdot\rangle$ is a Hilbert space.

Next, we define the unbounded operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ by

$$Ay(t) := \begin{pmatrix} b \\ a_{xx} \\ -\left(\gamma a(1) + a_x(1) + \frac{\epsilon}{m}\eta\right) \end{pmatrix},\ y \in D(A), \tag{24}$$

where $D(A) := \{y(t) = (a,b,\eta) \in (H^2[0,1] \cap \mathcal{V}) \times \mathcal{V} \times \Re;\ \eta = mb(1)\}$. Using (24) it then follows that (22) can be rewritten in the form

$$\dot{y} = Ay, \tag{25}$$

$$y(0) = \Phi, \tag{26}$$

where $\dot{y} = \frac{dy(t)}{dt}$, and $\Phi := \begin{pmatrix} \phi \\ \psi \\ \eta(0) \end{pmatrix}$.

Theorem 1 *Operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ defined by (24) generates a C_o semigroup of contractions $T(t)$ on the energy space $\mathcal{H}$.*

Proof

According to the Lumer-Phillips Theorem, see Goldstein (1985) page 26, it is sufficient to show that A is a m-dissipative operator. First, take any $y = (a, b, \eta) \in D(A)$ and then a straightforward computation gives $\langle Ay, y\rangle = -\frac{\epsilon}{m^2}\eta^2 \leq 0$. So it follows that the operator A is dissipative. Then we have to prove that the system $(I - A)y = y_o$ is uniquely solvable for any given $y_o = (a_o, b_o, \eta_o) \in \mathcal{H}$. Observe that $(I - A)y = y_o$ is equivalent with

$$a - b = a_o, \quad b - a_{xx} = b_o, \tag{27}$$

$$\eta + \gamma a(1) + a_x(1) + \frac{\epsilon}{m}\eta = \eta_o. \tag{28}$$

Eliminating b from (27) and using $\eta(t) = mu_t(1, t)$ we obtain

$$a - a_{xx} = a_o + b_o \in L^2(0, 1), \tag{29}$$

$$a(0) = 0, \quad (m + \epsilon + \gamma)a(1) + a_x(1) = \eta_o + (\epsilon + m)a_o(1). \tag{30}$$

The second order ODE (29) subject to (30) has a unique solution $a \in H^2(0, 1) \cap V$. From (28) the function η can be found. By this construction we found that $y = (a, b, \eta) \in D(A)$. So, the proof of the theorem now follows directly from the Lumer-Phillips Theorem. $\Box$

If A is a linear operator on $\mathcal{H}$ generating the C_o semigroup $T(t)$ and if the function y_o is in $D(A)$ then we can show that $T(t)y_o$ is in $D(A)$. Moreover, we have the following lemma (see Renardy and Rogers (1993) page 398).

Lemma 1 *Let A be the infinitesimal generator of the C_o semigroup $T(t)$. Then for any $f \in D(A)$ we have $T(t)f \in D(A)$ and the function $[0, \infty) \ni t \mapsto T(t)f \in \mathcal{H}$ is differentiable. In fact, $\frac{d}{dt}T(t)f = AT(t)f = T(t)Af$.*

For $y_o \in D(A)$ we define $(a, b, \eta) = y(t) := T(t)y_o$. Applying lemma 1 we find that $a \in C^2\left(R^+; L^2[0, 1]\right) \cap C\left(R^+; \mathcal{V} \cap H^2(0, 1)\right)$. So $y(t) = T(t)y_o$ is a strong solution of (25) - (26) for all $y_o \in D(A)$. But for $y_o \in D(A^2)$, applying lemma 1 twice we have

$$\begin{pmatrix} a_{tt} \\ b_{tt} \\ \eta_{tt} \end{pmatrix} = A \begin{pmatrix} b \\ a_{xx} \\ -\gamma a(1) - a_x(1) - \frac{\epsilon}{m}\eta \end{pmatrix} = \begin{pmatrix} a_{xx} \\ b_{xx} \\ r \end{pmatrix}, \tag{31}$$

where $r = -\gamma b(1) - b_x(1) + \frac{\epsilon\gamma}{m}a(1) + \frac{\epsilon}{m}a_x(1) + \left(\frac{\epsilon}{m}\right)^2 \eta$. From (31) and the definition of $D(A)$ we obtain $t \longmapsto a(t) \in C^1\left(\Re^+, H^2 \cap \mathcal{V}\right)$ and $t \longmapsto a_{xx} \in \mathcal{V}$. On the other hand, we also have $t \longmapsto a(t) \in C^2\left(\Re^+, \mathcal{V}\right)$. Then it follows that

$$a \in C^2\left(R^+; \mathcal{V}\right) \cap C^1\left(R^+; \mathcal{V} \cap H^2(0, 1)\right) \cap C\left(\Re^+; H^3(0, 1) \cap \mathcal{V}\right). \tag{32}$$

So from (32), for all $y_o \in D(A^2)$, $y(t) = T(t)y_o$, we have the equivalence between problem (1) - (5) and problem (25) - (26). So, the following theorem has now been established.

Theorem 2 *Let $\Phi \in D(A^2)$, then the initial - boundary value problem (1) - (5) and the initial value problem (25) - (26) are equivalent.*

Next, we will show that the solution of the initial - boundary value problem (1) - (5) depends continuously on the initial values. Let $\hat{y}(t)$ satisfy (25) with the initial values $\hat{y}(0) = \hat{\Phi}$ where $\hat{\Phi}= \begin{pmatrix} \hat{\phi} \\ \hat{\psi} \\ \hat{\eta}(0) \end{pmatrix}$, $(\hat{\phi}, \hat{\psi}) \in \left(C^2[0,1] \cap \mathcal{V}\right) \times \mathcal{V}$. Now, we approximate the difference between $y(t)$ and $\hat{y}(t)$, as follows; $\|y(t) - \hat{y}(t)\|_{\mathcal{H}} \leq \left\|T(t)(\Phi - \hat{\Phi})\right\|_{\mathcal{H}} \leq \left\|\Phi - \hat{\Phi}\right\|_{\mathcal{H}}$ for all $t \geq 0$. This means that small differences between the initial values cause small differences between the solution $y(t)$ and $\hat{y}(t)$ for all $t \geq 0$. We observe that if we take $\phi(x) \in H^3(0,1)$, $\phi(0) = \phi''(0) = 0$ and $\psi(x) \in H^2(0,1) \cap \mathcal{V}$ then we have Φ in the domain A^2. So, we can now formulate the following theorem on the well-posedness of the initial-boundary value problem (1) - (5).

Theorem 3 *Suppose $\phi(x) \in H^3(0,1)$, $\phi(0) = \phi''(0) = 0$ and $\psi(x) \in H^2(0,1) \cap \mathcal{V}$, $\psi(o) = o$, then problem (1) - (5) has a unique and twice continuously differentiable solution for $x \in [0,1]$ and $t \geq 0$. Moreover, this solution depends continuously on the initial values.*

5. THE CONSTRUCTION OF A FORMAL APPROXIMATION

In this section, an approximation of the solution of the initial-boundary value problem (1) - (5) will be constructed using a two-timescales perturbation method. If we expand the solution in a Taylor series with respect to ϵ straightforwardly , that is,

$$u(x,t) = u_o(x,t) + \epsilon u_1(x,t) + \epsilon^2 u_2(x,t) + \epsilon^3 \cdots , \tag{33}$$

the approximation of the solution of the problem will contain secular terms. From the energy integral in section 3, we know that the solution is bounded. So, the secular terms should be avoided. That is why a two-timescales perturbation method (as described in Kevorkian and Cole, 1981; Nayfeh, 1973) will be applied. Using such a two - timescales perturbation method the function $u(x,t)$ is supposed to be a function of x, t and $\tau = \epsilon t$. For that reason, we put

$$u(x,t) = w(x,t,\tau;\epsilon). \tag{34}$$

Using (34) the initial - boundary value problem (1) - (5) becomes

$$\begin{aligned}
w_{tt} + 2\epsilon w_{t\tau} + \epsilon^2 w_{\tau\tau} - w_{xx} &= 0, \quad 0 < x < 1,\, t > 0,\, \tau > 0, && (35)\\
w(0,t,\tau;\epsilon) &= 0, \quad t > 0,\, \tau > 0, && (36)\\
m w_{tt}(1,t,\tau;\epsilon) + \gamma w(1,t,\tau;\epsilon) + w_x(1,t,\tau;\epsilon) &= -\epsilon(w_t(1,t,\tau;\epsilon) + 2mw(1,t,\tau;\epsilon)_{t\tau}) \\
-\epsilon^2(w_\tau(1,t,\tau;\epsilon) &+ m w_{\tau\tau}(1,t,\tau;\epsilon)),\ t > 0,\ \tau > 0, && (37)\\
w(x,0,0;\epsilon) &= \phi(x),\ 0 < x < 1, && (38)\\
w_t(x,0,0;\epsilon) + \epsilon w_\tau(x,0,0;\epsilon) &= \psi(x),\ 0 < x < 1, && (39)
\end{aligned}$$

with

$$\phi(x) \in C^5\left([0,1];\Re\right), \quad \psi(x) \in C^4\left([0,1];\Re\right), \tag{40}$$

and

$$\begin{aligned}
\phi(0) = \phi''(0) = \phi^{iv}(0) &= 0, && (41)\\
m\phi''(1) + \gamma\phi(1) + \phi'(1) &= m\phi^{(iv)}(1) + \gamma\phi''(1) + \phi'''(1) = 0, && (42)\\
\psi(0) = \psi''(0) &= 0, && (43)\\
\psi(1) = m\psi''(1) + \psi'(1) &= 0. && (44)
\end{aligned}$$

Using a two time-scales perturbation method it is usually assumed that not only the solution $u(x,t)$ will depend on two time-scales but that also $u(x,t) = w(x,t,\tau;\epsilon)$ can be approximated by the formal expansion

$$u_o(x,t,\tau) + \epsilon u_1(x,t,\tau) + \cdots . \quad (45)$$

It is reasonable to assume this solution form because the PDE and the BCs depend analytically on ϵ. Substituting (45) into (35) - (39) and after equating the coefficients of like powers in ϵ, it follows that u_o has to satisfy

$$\begin{aligned}
u_{o_{tt}} - u_{o_{xx}} &= 0, \quad 0 < x < 1,\ t > 0,\ \tau > 0, & (46)\\
u_o(0,t,\tau) &= 0, \quad t > 0,\ \tau > 0, & (47)\\
mu_{o_{tt}}(1,t,\tau) + \gamma u_o(1,t,\tau) + u_{o_x}(1,t,\tau) &= 0,\ t > 0,\ \tau > 0, & (48)\\
u_o(x,0,0) &= \phi(x), \quad 0 < x < 1, & (49)\\
u_{o_t}(x,0,0) &= \psi(x), \quad 0 < x < 1. & (50)
\end{aligned}$$

The solution of (46) - (50) follows from section 2, yielding

$$u_o(x,t,\tau) = \sum_{n=1}^{\infty} \left(A_n(\tau)\sin(\sqrt{\lambda_n}t) + B_n(\tau)\cos(\sqrt{\lambda_n}t) \right) \sin(\sqrt{\lambda_n}x), \quad (51)$$

where $A_n(0)$ and $B_n(0)$ are given by

$$\begin{aligned}
A_n(0) &= \frac{\int_0^1 [1 + m\delta(x-1)]\psi(x)\sin(\sqrt{\lambda_n}x)dx}{\int_0^1 [1 + m\delta(x-1)]\sin^2(\sqrt{\lambda_n}x)dx}, & (52)\\
B_n(0) &= \frac{1}{\sqrt{\lambda_n}} \frac{\int_0^1 [1 + m\delta(x-1)]\phi(x)\sin(\sqrt{\lambda_n}x)dx}{\int_0^1 [1 + m\delta(x-1)]\sin^2(\sqrt{\lambda_n}x)dx}, & (53)
\end{aligned}$$

and where λ_n is given by (15). It follows from (40) - (44) and (15) that

$$|A_n(0)| \le \frac{2C_1}{\lambda_n^{5/2}} \quad \text{and} \quad |B_n(0)| \le \frac{2C_o}{\lambda_n^{5/2}}, \quad (54)$$

where C_o and C_1 are given by

$$C_o = \max_{1 \le n < \infty} \left\{ \left| \frac{\gamma}{\sqrt{\lambda_n}} \phi^{(iv)}(1)\sin(\sqrt{\lambda_n}) - \int_0^1 \phi^{(v)}(x)\cos(\sqrt{\lambda_n}x)dx \right| \right\}, \quad (55)$$

and

$$C_1 = \max_{1 \le n < \infty} \left\{ \left| -(\gamma\psi''(1) + \psi'''(1))\sin(\sqrt{\lambda_n}) + \int_0^1 \psi^{(iv)}(x)\sin(\sqrt{\lambda_n}x)dx \right| \right\}, \quad (56)$$

respectively. The $O(\epsilon)$ - problem for u_1 is given by

$$\begin{aligned}
u_{1_{tt}} - u_{1_{xx}} &= -2u_{o_{t\tau}},\ 0 < x < 1,\ t > 0, \tau > 0, & (57)\\
u_1(0,t,\tau) &= 0, \quad t > 0,\ \tau > 0, & (58)\\
mu_{1_{tt}}(1,t,\tau) + \gamma u_1(1,t,\tau) + u_{1_x}(1,t,\tau) &= -2mu_{o_{t\tau}}(1,t,\tau) - u_{o_t}(1,t,\tau), & (59)\\
&\quad t > 0,\ \tau > 0, &\\
u_1(x,0,0) &= 0, \quad 0 < x < 1, & (60)\\
u_{1_t}(x,0,0) &= -u_{o_\tau}(x,0,0), \quad 0 < x < 1. & (61)
\end{aligned}$$

To solve (57) - (61) the eigenfunction expansion approach will be used. Making boundary conditions homogeneous is the usual way to solve initial - boundary value problems when the inhomogeneous boundary conditions are of classical type (that is, are of Dirichlet, Neumann, or of Robin type). For the non-classical boundary condition at $x = 1$ this approach turns out to be not applicable. When we apply the eigenfunction expansion to solve the initial-boundary value problem (57) - (61) the left-hand side of (57) at $x = 1$ and that of (59) are of the same form. So, to solve the problem correctly the right-hand side of (57) at $x = 1$, and that of (59) should match, that is, should be proportional. To obtain this matching we introduce the following transformation

$$u_1(x,t,\tau) = xg(t,\tau) + v(x,t,\tau). \tag{62}$$

Substituting (62) into (57) - (61) we obtain

$$\begin{aligned}
v_{tt} - v_{xx} &= -2u_{o_{tt}} - xg_{tt},\ 0 < x < 1,\ t > 0, \tau > 0, && (63)\\
v(0,t,\tau) &= 0,\ t > 0, \tau > 0, && (64)\\
mv_{tt}(1,t,\tau) + \gamma v(1,t,\tau) + v_x(1,t,\tau) &= -2mu_{o_{t\tau}}(1,t,\tau) - u_{o_t}(1,t,\tau) - mg_{tt}(t,\tau) && (65)\\
&\quad -(\gamma+1)g(t,\tau),\ t > 0, \tau > 0, &&\\
v(x,0,0) &= -xg(0,0),\ 0 < x < 1, && (66)\\
v_t(x,0,0) &= -u_{o_\tau}(x,0,0) - xg_t(0,0),\ 0 < x < 1. && (67)
\end{aligned}$$

To solve (63) - (67) $v(x,t,\tau)$ is written in the eigenfunction expansion

$$v(x,t,\tau) = \sum_{n=1}^{\infty} v_n(t,\tau)\sin(\sqrt{\lambda_n}x). \tag{68}$$

Substituting (68) into (63) and (65) and taking the limit for $x = 1$ we obtain

$$\sum_{n=1}^{\infty} \left(v_{n_{tt}}(t,\tau) + \lambda_n v_n(t,\tau)\right)\sin\left(\sqrt{\lambda_n}.1\right) = -2u_{o_{t\tau}}(1,t,\tau) - g_{tt}(t,\tau) \tag{69}$$

and

$$\sum_{n=1}^{\infty}\left((mv_{n_{tt}}(t,\tau) + \gamma v_n(t,\tau))\sin(\sqrt{\lambda_n}) + \sqrt{\lambda_n}v_n(t,\tau)\cos(\sqrt{\lambda_n})\right) = -2mu_{o_{t\tau}}(1,t,\tau) - u_{o_t}(1,t,\tau) - mg_{tt}(t,\tau) - (\gamma+1)\,g(t,\tau) \tag{70}$$

respectively. From (15) it follows that $m\lambda_n = \gamma + \sqrt{\lambda_n}\cot\left(\sqrt{\lambda_n}\right)$, and so m times the left-hand side of (69) is equal to that of (70). And so, m times the right-hand side of (69) should be equal to that of (70). It then follows that

$$g(t,\tau) = -\frac{1}{\gamma+1}u_{o_t}(1,t,\tau). \tag{71}$$

The initial - boundary value problem (63) - (67) now becomes

$$\begin{aligned} v_{tt} - v_{xx} &= -2u_{o_{t\tau}} + \tfrac{1}{\gamma+1} \; x u_{o_{ttt}}(1,t,\tau),\ 0<x<1,\ t>0, \tau>0, && (72)\\ v(0,t,\tau) &= 0,\ t>0, \tau>0, && (73)\\ mv_{tt}(1,t,\tau)+\gamma v(1,t,\tau)+v_x(1,t,\tau) &= -2mu_{o_{t\tau}}(1,t,\tau)+m\frac{1}{\gamma+1}u_{o_{ttt}}(1,t,\tau), \\ & \quad t>0, \tau>0, && (74)\\ v(x,0,0) &= \frac{\psi(1)}{\gamma+1}x,\ 0<x<\pi, && (75)\\ v_t(x,0,0) &= -u_{o_\tau}(x,0,0)+\frac{\phi''(1)}{\gamma+1}x,\ 0<x<\pi. && (76) \end{aligned}$$

It should be observed that if m is equal to zero then the boundary condition at $x = 1$ becomes a classical boundary condition. From (74) it can readily be seen that in that case the boundary condition (59) at $x = 1$ becomes an homogeneous one after the transformation (62). When we expand x in a Fourier series, that is, $x = \sum_{n=1}^{\infty} c_n \sin(\sqrt{\lambda_n}x)$, where c_n is given by

$$c_n = \frac{\int_0^1 x[1+m\delta(x-1)]\sin(\sqrt{\lambda_n}x)dx}{\int_0^1 [1+m\delta(x-1)]\sin^2(\sqrt{\lambda_n}x)dx} = \frac{2(\gamma+1)\sin(\sqrt{\lambda_n})}{\lambda_n+(m\lambda_n+\gamma)\sin^2(\sqrt{\lambda_n})} \qquad (77)$$

the initial - boundary value problem (72) - (76) can now be solved by substituting (68) into the partial differential equation (72), yielding

$$\begin{aligned} v_{n_{tt}} &+ \lambda_n v_n = (2\sqrt{\lambda_n}A_n' + \frac{c_n}{\gamma+1}\lambda_n^{3/2}\sin(\sqrt{\lambda_n})A_n)\sin(\sqrt{\lambda_n}t)\\ &- (2\sqrt{\lambda_n}B_n' + \frac{c_n}{\gamma+1}\lambda_n^{3/2}\sin(\sqrt{\lambda_n})B_n)\cos(\sqrt{\lambda_n}t)\\ &+ \frac{c_n}{\gamma+1}\sum_{\substack{p=1\\ p\neq n}}^{\infty}\lambda_p^{3/2}\sin(\sqrt{\lambda_p})(A_p\sin(\sqrt{\lambda_p}t)-B_p\cos(\sqrt{\lambda_p}t)). \qquad (78) \end{aligned}$$

Observe that $v(x,t,\tau)$ now automatically satisfies the boundary conditions at $x = 0$ and $x = 1$. In order to remove secular terms, it now easily follows from (78) that A_n and B_n have to satisfy

$$\begin{aligned} A_n' + \frac{c_n}{2(\gamma+1)}\lambda_n\sin(\sqrt{\lambda_n})A_n &= 0, && (79)\\ B_n' + \frac{c_n}{2(\gamma+1)}\lambda_n\sin(\sqrt{\lambda_n})B_n &= 0. && (80) \end{aligned}$$

The solution of (79) - (80) is given by

$$\begin{aligned} A_n(\tau) &= A_n(0)\exp(-\alpha_n\tau), && (81)\\ B_n(\tau) &= B_n(0)\exp(-\alpha_n\tau), && (82) \end{aligned}$$

where $\alpha_n = \frac{\lambda_n\sin^2(\sqrt{\lambda_n})}{\lambda_n+(m\lambda_n+\gamma)\sin^2(\sqrt{\lambda_n})} > 0$. From (54) and (81) - (82) it follows that the infinite series representation (51) for u_o is twice continuously differentiable with respect to x and t, and infinitely many times with respect to τ. From (15) it follows that $\sqrt{\lambda_n} \to (n-1)\pi$ as $n \to \infty$. So, α_n tends to 0 as n tends to ∞. From (81) and (82) it then follows that u_o is stable but not uniform. From (79) - (80) $v_n(t,\tau)$ in (78) can now be determined, yielding

$$\begin{aligned} v_n(t,\tau) &= D_n(\tau)\cos(\sqrt{\lambda_n}t) + E_n(\tau)\sin(\sqrt{\lambda_n}t) \\ &+ \sum_{\substack{p=1\\p\neq n}}^{\infty} \frac{1}{\gamma+1}\frac{\lambda_p^{3/2}}{\lambda_p-\lambda_n}\sin(\sqrt{\lambda_p})(-A_p(\tau)\sin(\sqrt{\lambda_p}t) + B_p(\tau)\cos(\sqrt{\lambda_n}t)), \quad (83) \end{aligned}$$

where $D_n(\tau)$ and $E_n(\tau)$ are still arbitrary functions which can be used to avoid secular terms in $u_2(x,t,\tau)$. From (75) and from (76) it follows that

$$\begin{aligned} D_n(0) &= \frac{1}{\beta_n}\int_0^1 [1+m\delta(x-1)]\left[\frac{\psi(1)}{\gamma+1}x\right]\sin(\sqrt{\lambda_n}x)dx \quad (84) \\ &\quad -\frac{c_n}{\gamma+1}\sum_{\substack{p=1\\p\neq n}}^{\infty}\frac{\lambda_p^{3/2}}{\lambda_p-\lambda_n}\sin(\sqrt{\lambda_p})B_p(0), \\ \sqrt{\lambda_n}E_n(0) &= \frac{1}{\beta_n}\int_0^1 [1+m\delta(x-1)]\left[\frac{\phi''(1)}{\gamma+1}x\right]\sin(\sqrt{\lambda_n}x)dx \quad (85) \\ &\quad +\alpha_n B_n(0) + \frac{c_n}{\gamma+1}\sum_{\substack{p=1\\p\neq n}}^{\infty}\frac{\lambda_p^2}{\lambda_p-\lambda_n}\sin(\sqrt{\lambda_p})A_n(0), \end{aligned}$$

where $\beta_n = \int_0^1 [1+m\delta(x-1)]\sin^2(\sqrt{\lambda_n}x)dx = \frac{1}{2}\left[1+\left[\frac{m\lambda_n+\gamma}{\lambda_n}\right]\sin^2\sqrt{\lambda_n}\right] \geq \frac{1}{2}$. Elementarily it can be shown that $|D_n(0)| \leq \frac{C_2}{\lambda_n^2}$ and $E_n(0) \leq \frac{C_3}{\lambda_n^2}$, where C_2 and C_3 are constants. The solution $u_1(x,t,\tau)$ of (57) - (61) now easily follows from (62), (68), (71), and (83), yielding

$$u_1(x,t,\tau) = \sum_{n=1}^{\infty}\left(v_n(t,\tau) + \frac{c_n}{\gamma+1}\sum_{p=1}^{\infty}H_p(t,\tau)\right)\sin(\sqrt{\lambda_n}x), \quad (86)$$

where $H_p(t,\tau) = \sqrt{\lambda_p}\sin(\sqrt{\lambda_p})\left(B_p(\tau)\sin(\sqrt{\lambda_p}t) - A_p(\tau)\cos(\sqrt{\lambda_p}t)\right)$, where v_n is given by (83), where $A_n(\tau)$ and $B_n(\tau)$ are given by (81) and (82), and where c_n is given by (77). It should be observed that $u_1(x,t,\tau)$ still contains infinitely many undetermined functions $D_n(\tau)$ and $E_n(\tau)$, $n = 1,2,3,\cdots$. These functions can be used to avoid secular terms in the function $u_2(x,t,\tau)$. At this moment, however, we are not interested in the higher order approximations. For that reason we will take $D_n(\tau) = D_n(0)$ and $E_n(\tau) = E_n(0)$. So far we have constructed a formal approximation $\bar{u}(x,t) = u_o(x,t,\tau) + \epsilon u_1(x,t,\tau)$ of $u(x,t)$, where $u_o(x,t,\tau)$ and $u_1(x,t,\tau)$ are twice continuously differentiable with respect to x and t, and infinitely many times with respect to τ.

6. ON THE ASYMPTOTIC VALIDITY OF FORMAL APPROXIMATIONS

In this section, we study the asymptotic validity of formal approximations on $0 \leq t \leq L_o\epsilon^{-1}$ and $0 \leq x \leq 1$, where L_o is an ϵ-independent constant. We will show that a formal approximation of the solution is indeed an asymptotic approximation if the approximation satisfies the PDE, and the ICs and BCs up to some, specified order in ϵ. In section 5 we have constructed the function

$$\bar{u}(x,t) = u_o(x,t,\tau) + \epsilon u_1(x,t,\tau). \quad (87)$$

This function is a so-called formal approximation of the solution and this function satisfies the following initial-boundary value problem

$$\begin{aligned} \bar{u}_{tt} - \bar{u}_{xx} &= \epsilon^2 F(x,t;\epsilon), \;\; 0 < x < 1, \;\; t > 0, \tau > 0, && (88)\\ \bar{u}(0,t) &= 0, \qquad t > 0, \tau > 0, && (89)\\ m\bar{u}_{tt}(1,t) + \gamma\bar{u}(1,t) + \bar{u}_x(1,t) + \epsilon\bar{u}_t(1,t) &= \epsilon^2 R(t,\tau;\epsilon), \;\; t > 0, \tau > 0, && (90)\\ \bar{u}(x,0) &= \phi(x), \quad 0 < x < 1, && (91)\\ \bar{u}_t(x,0) &= \psi(x) + \epsilon^2 u_{1_\tau}(x,0,0), \;\; 0 < x < 1, && (92) \end{aligned}$$

where

$$\begin{aligned} F(x,t;\epsilon) &= u_{o_{\tau\tau}}(x,t,\tau) + 2u_{1_{t\tau}}(x,t,\tau) + \epsilon u_{1_{\tau\tau}}(x,t,\tau),\\ R(t,\tau;\epsilon) &= mu_{o_{\tau\tau}}(1,t,\tau) + 2mu_{1_{t\tau}}(1,t,\tau) + u_{o_\tau}(1,t,\tau) + u_{1_t}(1,t,\tau)\\ &\quad +\epsilon m u_{1\tau\tau}(1,t,\tau) + \epsilon u_{1\tau}(1,t,\tau), \end{aligned}$$

To prove the asymptotic validity of $\bar{u}(x,t,\tau)$ we define some auxiliary functions, $\bar{a}(t) = \bar{u}(\bullet,t)$, $\bar{b}(t) = \bar{u}_t(\bullet,t)$, and $\bar{\eta}(t) = m\bar{u}_t(1,t)$. We also denote $\bar{a}$, $\bar{b}$ and $\bar{\eta}$ for $\bar{a}(t)$, $\bar{b}(t)$ and $\bar{\eta}(t)$, respectively. By differentiating these functions with respect to t we obtain

$$\begin{pmatrix} \bar{a}_t \\ \bar{b}_t \\ \bar{\eta}_t \end{pmatrix} = \begin{pmatrix} \bar{b} \\ \bar{a}_{xx} \\ -\gamma\bar{a}(1) - \bar{a}_x(1) - \frac{\epsilon}{m}\eta \end{pmatrix} + \epsilon^2 \begin{pmatrix} 0 \\ F(\bullet,t;\epsilon) \\ R(1,t;\epsilon) \end{pmatrix}. \tag{93}$$

We also define the same operator A as in section 4, i.e. $A\bar{y} = \begin{pmatrix} \bar{b} \\ \bar{a}_{xx} \\ -\gamma\bar{a}(1) - \bar{a}_x(1) - \frac{\epsilon}{m}\bar{\eta} \end{pmatrix}$, where $\bar{y} = \begin{pmatrix} \bar{a} \\ \bar{b} \\ \bar{\eta} \end{pmatrix}$. It then follows that (93) can be written as

$$\begin{aligned} \frac{d\bar{y}}{dt} &= A\bar{y} + \epsilon^2\Theta(t), && (94)\\ \bar{y}(0) &= \bar{\Phi}, && (95) \end{aligned}$$

where $\Theta(t) = \begin{pmatrix} 0 \\ F(\bullet,t;\epsilon) \\ R(1,t;\epsilon) \end{pmatrix}$, and $\bar{\Phi} = \Phi + \epsilon^2 \begin{pmatrix} 0 \\ u_{1\tau}(\bullet,0,0) \\ m\,(u_{1\tau}(1,0,0)) \end{pmatrix}$.

From the convergence and differentiability properties of the infinite series representations for u_o and u_1 it follows that Θ and $\bar{\Phi} - \Phi$ are bounded, that is, there are two constants M_0 and M_1 such that $\|\Theta(t)\|_{\mathcal{H}} \le M_0$ and $\left\|\bar{\Phi} - \Phi\right\|_{\mathcal{H}} \le \epsilon^2 M_1$. The solution of the initial value problem (94) - (95) is given by

$$\bar{y}(t) = T(t)\bar{\Phi} + \epsilon^2 \int_0^t T(t-s)\Theta(s)ds, \tag{96}$$

where $T(t)$ is defined as in section 4. For $0 \le t \le L_o\epsilon^{-1}$ and $0 \le x \le 1$, we can now estimate the difference between y and $\bar{y}$

$$\|y(t) - \bar{y}(t)\|_{\mathcal{H}} = \left\| T(t)\left(\Phi - \bar{\Phi}\right) - \epsilon^2 \int_0^t T(t-s)\Theta(s)ds \right\|_{\mathcal{H}} \le \epsilon(\epsilon M_1 + L_o M_0). \tag{97}$$

We can conclude from (97) that $y(t) - \bar{y}(t) = O(\epsilon)$ on a timescale of order $\frac{1}{\epsilon}$. From this it easily follows that $u(x,t) - (u_o(x,t,\tau) + \epsilon u_1(x,t,\tau)) = O(\epsilon)$ and $u(x,t) - u_o(x,t,\tau) = O(\epsilon)$ on $0 \le t \le L_o\epsilon^{-1}$ and $0 \le x \le 1$. And so we obtained the asymptotic validity of the formal approximations.

7. SUMMARY

In this paper an initial - boundary value problem for a weakly damped string has been considered. It can be shown that (using a semigroup approach) the initial-boundary value problem (1) - (5) is well-posed for $0 \leq x \leq 1$ and $t \geq 0$. Using an energy integral it can also be shown that the solution is bounded. The construction of the approximation is far from being elementary. For instance it is not possible to solve (57) - (61) in the classical way by making the boundary condition at $x = 1$ homogeneous. This is due to the non-classical boundary condition at $x = 1$. It can only be done by balancing or matching the right-hand side of (57) and that of (59) by transforming u in an appropriate way. It also should be noted that the way to solve the wave equation with a non-classical boundary condition (using the eigenfunction expansion as we have done in section 5) is an extension of the classical way to solve such problems. Finally, we proved that the formal approximation is an asymptotic one on a time-scale of order ϵ^{-1}.

Although we did not consider external forces (leading to an inhomogeneous PDE) these problems can be solved in a similar way using the balancing or matching procedure as given in section 5. Actually by considering (57) - (61) we have solved an inhomogeneous problem. The results presented in this paper most likely can be extended to weakly nonlinear pdes with non-classical boundary conditions.

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