ON THE CHARACTERISTIC LAYER OF A LINEAR DAMPED SYSTEM

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Abstract. In this paper initial-boundary value problems for a string (a wave) equation are considered. One end of the string is assumed to be fixed and the other end of the string is attached to a dashpot system, where the damping generated by the dashpot system is assumed to be small, and is assumed to be proportional to the vertical and the angular velocity of the string in the endpoint. This problem can be regarded as a rather simple model describing oscillations of flexible structures such as for instance overhead power transmission lines. The method of separation of variables will be used to construct asymptotic approximations of the solution.

keywords: *Characteristic layer, boundary damping, asymptotics, perturbation method.*

1. Introduction

In this paper the following initial-boundary value problem will be considered.

$$u_{tt} - u_{xx} + p^2 u = 0,\ 0 < x < \pi,\ t > 0, \tag{1.1}$$

$$u(0,t) = 0,\ t \geq 0, \tag{1.2}$$

$$u_x(\pi,t) = -\epsilon\left(\beta u_{xt}(\pi,t) + \alpha u_t(\pi,t)\right),\ t \geq 0, \tag{1.3}$$

$$u(x,0) = \phi(x),\ 0 < x < \pi, \tag{1.4}$$

$$u_t(x,0) = \psi(x),\ 0 < x < \pi, \tag{1.5}$$

where ϵ is a small, positive parameter with $0 < \epsilon \ll 1$, where α, β (the damping coefficients of the dashpot which are assumed to be small), $p^2 \geq 0$ (for instance the stiffness of the stays of the bridge) are positive constants, and where ϕ and ψ are the initial displacement and the initial velocity of the string respectively. The functions ϕ and ψ are assumed to be sufficiently smooth and to be of order one. This problem can be regarded as a rather simple model describing oscillations of flexible structures such as for instance overhead power transmission lines. To derive this model we refer to [1, 2]

2. Approximation of eigenvalue

In this section an approximation of the solution of the initial-boundary value problem (1.1)-(1.5) will be constructed for arbitrary α and β. To construc the approximation the method of separation of variables will be used. An alternative way to approximate the solution the method of multiple scales (as described in [3]) can also be used. The former approach of course can only be applied to linear problems, whereas the latter method can also be applied to weakly nonlinear problems.

2.1. Separation of variables.

Since the initial-boundary value problem (1.1)-(1.5) is linear the method of separation of variables can be used to solve problem (1.1)-(1.5) approximately. In this section we will shortly outline how nontrivial solutions of the boundary value problem (1.1)-(1.3) can be obtained in the form $X(x)T(t)$. By Substituting $X(x)T(t)$ into (1.1)-(1.3) the following eigenvalue problem is obtained

$$\frac{X''(x)}{X(x)} = \rho^2 = \frac{T''(t)}{T(t)} + p^2, \quad \rho \in \mathbb{C}, \quad 0 < x < \pi, \quad t \geq 0, \tag{2.6}$$

$$X(0) = 0, \tag{2.7}$$

$$X'(\pi)T(t) = -\left(\epsilon\beta X'(\pi) + \epsilon\alpha X(\pi)\right)T'(t), \tag{2.8}$$

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where $\rho \in \mathbb{C}$ is a separation parameter. From (2.6) and (2.7) it follows that $X(x) = A\sinh(\rho x)$ with A an arbitrary constant. By substituting $X(x) = A\sinh(\rho x)$ into (2.8) the following expression is obtained

$$\rho\cosh(\rho\pi)T(t) = -\left(\epsilon\beta\rho\cosh(\rho\pi) + \epsilon\alpha\sinh(\rho\pi)\right)T'(t). \tag{2.9}$$

It should be observed that for $\epsilon\beta\rho\cosh(\rho\pi)+\epsilon\alpha\sinh(\rho\pi) = 0$ only trivial solutions for the boundary-value problem (2.6)-(2.8) are obtained. By differentiating (2.9) with respect to t and by using (2.9) again it follows that

$$T''(t) = \frac{\rho^2\cosh^2(\rho\pi)}{\left(\epsilon\beta\rho\cosh(\rho\pi) + \epsilon\alpha\sinh(\rho\pi)\right)^2}T(t). \tag{2.10}$$

By substituting (2.10) into (2.6) it then follows that ρ has to satisfy

$$\rho^2 = \frac{\rho^2\cosh^2(\rho\pi)}{\left(\epsilon\beta\rho\cosh(\rho\pi) + \epsilon\alpha\sinh(\rho\pi)\right)^2} + p^2 \tag{2.11}$$

with $\rho \in \mathbb{C}$. Approximations of ρ can be obtained from (2.11) in the following way. First (2.11) is rewritten in

$$\left(\rho^2 - p^2\right)\left(\epsilon\beta\rho\cosh(\rho\pi) + \epsilon\alpha\sinh(\rho\pi)\right)^2 = \rho^2\cosh^2(\rho\pi), \tag{2.12}$$

and then the following two cases $\rho\cosh(\rho\pi) = 0 + O(\epsilon)$ and $\rho\cosh(\rho\pi) \neq 0 + O(\epsilon)$ are considered.

2.1.1. *The case* $\rho\cosh(\rho\pi) = 0+O(\epsilon)$. For this case we have to consider two subcases, that is, $\cosh(\rho\pi) = 0 + O(\epsilon)$ or $\rho = 0 + O(\epsilon)$. Dividing (2.12) by $\rho^2 - p^2$, putting $\rho = a + bi$ with $a, b \in \Re$, and then taking apart the real part and the imaginary part of the so-obtained equation, it follows that

$$X_0X_1 + Y_0Y_1 = \epsilon^2\left(X_5 - X_6 + X_4\right), \tag{2.13}$$

and

$$X_0Y_1 - X_1Y_0 = \epsilon^2\left(X_7 - X_8 + Y_4\right), \tag{2.14}$$

where

$$\begin{aligned}
X_0 &= \frac{\left(a^2-b^2\right)\left(a^2-b^2-p^2\right)+4a^2b^2}{\left(a^2-b^2-p^2\right)+4a^2b^2}, \quad Y_0 = \frac{2abp^2}{\left(a^2-b^2-p^2\right)+4a^2b^2},\\
X_1 &= \cos(2b\pi)\cosh(2a\pi)+1, \quad Y_1 = \sin(2b\pi)\sinh(2a\pi),\\
X_4 &= \beta^2\left(a^2-b^2\right)+\alpha^2, \quad Y_4 = 2ab\beta^2,\\
X_5 &= \cos(2b\pi)\left(\left[\beta^2\left(a^2-b^2\right)+\alpha^2\right]\cosh(2a\pi) + 2a\alpha\beta\sinh(2a\pi)\right),\\
X_6 &= 2b\beta\sin(2b\pi)\left(a\beta\sinh(2a\pi)+\alpha\cosh(2a\pi)\right),\\
X_7 &= 2b\beta\cos(2b\pi)\left(a\beta\cosh(2a\pi)+\alpha\sinh(2a\pi)\right),\\
X_8 &= \sin(2b\pi)\left(\left[\beta^2\left(a^2-b^2\right)+\alpha^2\right]\sinh(2a\pi) + 2a\alpha\beta\cosh(2a\pi)\right).
\end{aligned}$$

It is assumed that the eigenvalue $\rho = a + bi$ can be expanded in a power series in ϵ, that is,

$$a = a_0 + \epsilon a_1 + \epsilon^2 a_2 + \ldots, \tag{2.15}$$

$$b = b_0 + \epsilon b_1 + \epsilon^2 b_2 + \ldots. \tag{2.16}$$

To approximate ρ, (2.13) and (2.14) are then expanded in power series in ϵ. For the case $\cosh(\rho\pi) = 0+O(\epsilon)$ it follows that $\rho = a+ib = i(n-\frac{1}{2})+O(\epsilon)$. By substituting (2.15) and (2.16) (in this case $a_0 = 0$ and $b_0 = n - 1/2$ with $n \in \mathbb{Z}$) into (2.13)-(2.14) and equating the coefficients of ϵ^n for $n = 0, 1, 2, \ldots$ the following results will be obtained

$$\rho = \frac{\alpha}{\pi}\sqrt{1+\frac{p^2}{(n-\frac{1}{2})^2}}\epsilon + \left(n - \frac{1}{2} + \frac{\alpha}{\pi}\left(\beta(n-\frac{1}{2})\left(1+\frac{p^2}{(n-\frac{1}{2})^2}\right) + \frac{\alpha}{\pi}\frac{p^2}{(n-\frac{1}{2})^3}\right)\epsilon^2\right)i + O(\epsilon^3) \tag{2.17}$$

or

$$\rho = -\frac{\alpha}{\pi}\sqrt{1+\frac{p^2}{(n-\frac{1}{2})^2}}\epsilon + \left(n - \frac{1}{2} + \frac{\alpha}{\pi}\left(\beta(n-\frac{1}{2})\left(1+\frac{p^2}{(n-\frac{1}{2})^2}\right) + \frac{\alpha}{\pi}\frac{p^2}{(n-\frac{1}{2})^3}\right)\epsilon^2\right)i + O(\epsilon^3), \tag{2.18}$$

with $n \in \mathbb{Z}$.

For the case $\rho = 0 + O(\epsilon)$ it follows that $a_0 = b_0 = 0$. Again to approximate the eigenvalue ρ, (2.15) and (2.16) are substituted into (2.13)-(2.14). After expanding (2.13) and (2.14) into power series in ϵ and equating the coefficients of ϵ^n for $n = 0, 1, 2, \ldots$ the following result will be obtained $\rho = 0 + O(\epsilon^m)$ for all positive m. So, for $\rho = 0 + O(\epsilon)$ no eigenvalues are found.

Now we can approximate the solution for the case $\rho\cosh(\rho\pi) = 0+O(\epsilon)$. For instance, if we approximate the eigenvalue up to order ϵ, that is, $\rho = (n-1/2)i + \frac{\alpha}{\pi}\sqrt{1+\frac{p^2}{(n-\frac{1}{2})^2}}\epsilon$ or $\rho = (n-1/2)i - \frac{\alpha}{\pi}\sqrt{1+\frac{p^2}{(n-\frac{1}{2})^2}}\epsilon$ with $n \in \mathbb{Z}$. It follows from (2.6) that $u(x,t)$ for the case $\rho\cosh(\rho\pi) = 0 + O(\epsilon)$ is approximated by

$$\begin{aligned}(2.19)\qquad & \exp\left(-\frac{\alpha}{\pi}\epsilon t\right)\sum_{n=1}^{\infty}\left[\exp\left(\frac{\alpha}{\pi}\sqrt{\left(1+\frac{p^2}{(n-\frac{1}{2})^2}\right)}\epsilon x\right)\left[\bar{A}_n\left(\cos\left(\sqrt{\lambda_n}t\right)\cos\left(\left(n-\frac{1}{2}\right)x\right)+\right.\right.\right.\\
& \left.\sin\left(\sqrt{\lambda_n}t\right)\sin\left(\left(n-\frac{1}{2}\right)x\right)\right)+\bar{B}_n\left(\sin\left(\sqrt{\lambda_n}t\right)\cos\left(\left(n-\frac{1}{2}\right)x\right)-\right.\\
& \left.\left.\cos\left(\sqrt{\lambda_n}t\right)\sin\left(\left(n-\frac{1}{2}\right)x\right)\right)\right]-\\
& \exp\left(-\frac{\alpha}{\pi}\sqrt{\left(1+\frac{p^2}{(n-\frac{1}{2})^2}\right)}\epsilon x\right)\left[\bar{A}_n\left(\cos\left(\sqrt{\lambda_n}t\right)\cos\left(\left(n-\frac{1}{2}\right)x\right)-\right.\right.\\
& \left.\sin\left(\sqrt{\lambda_n}t\right)\sin\left(\left(n-\frac{1}{2}\right)x\right)\right)+\bar{B}_n\left(\sin\left(\sqrt{\lambda_n}t\right)\cos\left(\left(n-\frac{1}{2}\right)x\right)+\right.\\
& \left.\left.\left.\cos\left(\sqrt{\lambda_n}t\right)\sin\left(\left(n-\frac{1}{2}\right)x\right)\right)\right]\right],\end{aligned}$$

where $\lambda_n = p^2 + (n-1/2)^2$. The constants $\bar{A}_n$ and $\bar{B}_n$ can be determined by using the initial conditions (1.4) and (1.5).

2.1.2. *The case* $\rho\cosh(\rho\pi) \neq 0 + O(\epsilon)$. Dividing (2.12) by $\rho\cosh(\rho\pi)$ it follows that

$$(2.20)\qquad \left(1-\frac{p^2}{\rho^2}\right)(\epsilon\beta\rho + \epsilon\alpha\tanh(\rho\pi))^2 = 1.$$

It should be observed in this case that the first order approximation of ρ is proportional to $\frac{1}{\epsilon\beta}$. For that reason the eigenvalue ρ is approximated by $\rho = \frac{\rho_{-1}}{\epsilon} + \rho_0 + \epsilon\rho_1 + \epsilon^2\rho_2 + \ldots$. For the positive real part of the eigenvalue ρ it is clear that

$$(2.21)\qquad \lim_{\epsilon\downarrow 0}\tanh(\rho\pi) = 1.$$

So for $\epsilon \downarrow 0$ it then follows from (2.20) and (2.21) that

$$(2.22)\qquad \left(1-\frac{p^2}{\rho^2}\right)(\epsilon\beta\rho + \epsilon\alpha - 2\epsilon\alpha\frac{e^{-2\rho\pi}}{1+e^{-2\rho\pi}})^2 = 1.$$

Rescaling ρ by $\frac{\tilde{\rho}}{\epsilon}$, where $\tilde{\rho} = a + ib$ is of order one it follows that the real and the imaginary part of (2.22) have to satisfy

$$(2.23)\qquad \eta\left((\beta a + \epsilon\alpha)^2 - (\beta b)^2\right) - 2\beta b(\beta a + \epsilon\alpha)\theta = 1,$$

and

$$(2.24)\qquad \theta\left((\beta a + \epsilon\alpha)^2 - (\beta b)^2\right) + 2\beta b(\beta a + \epsilon\alpha)\eta = 0,$$

respectively,where $\eta = 1 - \frac{p^2(a^2-b^2)}{(a^2+b^2)^2}$ and $\theta = \frac{2abp^2}{(a^2+b^2)^2}$, and where terms which are exponentially small (that is, terms of order $\epsilon e^{\left(\frac{-2\tilde{\rho}\pi}{\epsilon}\right)}$) have been neglected. To approximate $\tilde{\rho}$ the power series representations (2.15) and (2.16) are used again. By substituting (2.15) and (2.16) into (2.23) and (2.24) the approximation of the eigenvalue ρ up to order ϵ^3 is given by

$$(2.25)\qquad \rho = \frac{1}{\epsilon\beta} - \frac{\alpha}{\beta} + \frac{\beta p^2}{2}\epsilon + \alpha\beta p^2\epsilon^2 + O(\epsilon^3).$$

When the eigenvalue ρ has a negative real part a compeletely similar analysis can be given, yielding

$$\rho = -\frac{1}{\epsilon\beta} + \frac{\alpha}{\beta} - \frac{\beta p^2}{2}\epsilon - \alpha\beta p^2\epsilon^2 + O(\epsilon^3). \tag{2.26}$$

As in section 2.1.1 a nontrivial solution $u(x,t)$ of the boundary value problem (2.6) - (2.8) for the case $\rho\cosh(\rho\pi) \neq 0 + O(\epsilon)$ can again be constructed (using (2.25) and (2.26), yielding

$$E\sinh\left(\left(1 - \epsilon\alpha + \frac{\beta^2 p^2\epsilon^2}{2}\right)\frac{x}{\epsilon\beta}\right)\exp\left(-\left(1-\epsilon\alpha\right)\frac{t}{\epsilon\beta}\right), \tag{2.27}$$

where E is a constant which follows by evaluating u_x at $x = \pi$ and $t = 0$. An approximation of the solution of the initial-boundary value problem (1.1)-(1.5) follows from (2.19) and (2.27).

It should be remarked from that (2.26) that the angular vilocity damper β only play a significant role in very small neighborhood of the boundary condition at $x = \pi$. The damper induces a small characteristic layer $x - t - \pi = O(\epsilon)$ in which the angle of the string tends to zero in a very short time. In other hand, it can be seen from (2.19) the boundary damping α acts to suppress oscillations in the entire string.

3. Conclusions and remarks

In this paper an initial-boundary value problem for a string equation has been considered. One end of the string is assumed to be fixed and the other end of the string is attached to a dashpot system, where the damping generated by the dashpot system is assumed to be small, and is assumed to be proportional to the vertical and the angular velocity of the string in the endpoint. It has been shown in section 4 that the presence of the term $-\epsilon\beta u_{xt}$ in the boundary condition at $x = \pi$ gives rise to a singularly perturbed problem, which in fact leads to a characteristic layer problem. This term (that is, $-\epsilon\beta u_{xt}$) is proportional to the angular velocity of the string in the endpoint at $x = \pi$. From the results it follows that due to this type of angular velocity damping the angle of the string at $x = \pi$ tends to zero in a very short time (that is, within times of $O(1)$). The vertical oscillations of the string are hardly influenced by this angular velocity damper. The vertical oscillations of the string also decrease to zero (due to the term $-\epsilon\alpha u_t$ in the boundary condition at $x = \pi$). In comparison with the method of multiple scales (as decribed in [3]) the method of separation of variables which is used in this paper is more effective and efficient. However, this approach can only be used for linear problems. Whereas the method of multiple scales can even be used for a large class of the problem even for nonlinear problems.

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