ON ASYMPTOTIC SOLUTIONS FOR A WAVE EQUATION WITH NON-CLASSICAL BOUNDARY CONDITIONS

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ABSTRACT. In this paper an initial-boundary value problem for a homogeneous string (or wave) equation is considered. One end of the string is assumed to be fixed and the other end of the string is attached to a spring-mass-dashpot system, where the damping generated by the dashpot is assumed to be small. This problem can be regarded as a simple model describing oscillations of flexible structures such as overhead power transmission lines. A multiple time-scales perturbation method will be used to construct asymptotic approximations of the solution. A semigroup approach will be used to show the asymptotic validity of formal approximations of the solution on long time-scales. Although the problem is linear the construction of these approximations is far from being elementary because of the complicated, non-classical boundary condition.

1. INTRODUCTION

There are a number of examples of flexible structures such as suspension bridges, overhead transmission lines and dynamically loaded helical springs that are subjected to oscillations due to different causes. Simple models which describe these oscillations can be expressed in initial-boundary value problem for wave equations like in [9,11,12] or for beam equations like in [2,4,5].

In most cases simple, classical boundary conditions are applied (such as in [5,12]) to construct approximations of the oscillations. For more complicated, non-classical boundary conditions (see for instance [2,9,10,11,]) it seems to be not possible to construct explicit approximations of the oscillations. In this paper we will study such an initial-boundary value problem with a non-classical boundary condition and we will construct asymptotic approximations of the solution, which are valid on a long time-scale. We will consider a string which is fixed at $x = 0$ and attached to a spring-mass-dashpot system at $x = 1$ (see also figure 1.1).

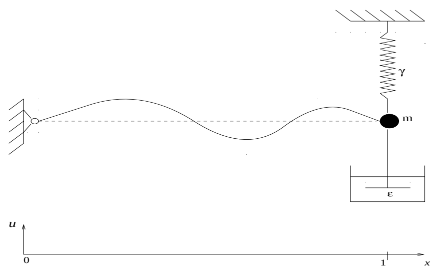

FIGURE 1.1. A simple model of a string fixed at $x = 0$ and attached to a spring-mass-dashpot system at $x = 1$.

It is assumed that ρ (the mass-density of the string), T (the tension in the string), $\tilde{m}$ (the mass in the spring-mass-dashpot system), $\tilde{\gamma}$ (the stiffness of the spring), and $\tilde{\epsilon}$ (the damping coefficient of the dashpot with $0 < \tilde{\epsilon} \ll 1$) are all positive constants. Furthermore, we only

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consider the vertical displacement $\tilde{u}(x,\tilde{t})$ of the string, where x is the place along the string, and $\tilde{t}$ is time. Gravity and other external forces are neglected.

After applying a simple rescaling in time and in displacement ($\tilde{t} = \sqrt{\frac{T}{c}}t$, $\tilde{u}(x,\tilde{t}) = u(x,t)$; putting $\tilde{m} = \rho.m$, $\tilde{\gamma} = \gamma.T$, and $\tilde{\epsilon} = \sqrt{Tc}\epsilon$) we obtain as a simple model for the oscillations of the string the following initial-boundary value problem

$$\begin{aligned} u_{tt} - u_{xx} &= 0, \;\; 0 < x < 1, \;\; t > 0, && (1.1)\\ u(0,t) &= 0, \;\; t \geq 0, && (1.2)\\ mu_{tt} + \gamma u + u_x &= -\epsilon u_t, \;\; x = 1, \;\; t \geq 0, && (1.3)\\ u(x,0) &= \phi(x), \;\; 0 < x < 1, && (1.4)\\ u_t(x,0) &= \psi(x), \;\; 0 < x < 1, && (1.5) \end{aligned}$$

with

$$\begin{aligned} &\phi(x) \in C^5\left([0,1];\Re\right), \quad \psi(x) \in C^4\left([0,1];\Re\right), && (1.6)\\ \phi(0) = \phi''(0) = \phi^{iv}(0) = \psi(0) = \psi''(0) &= 0 && (1.7)\\ m\phi''(1) + \gamma\phi(1) + \phi'(1) &= -\epsilon\psi(1), && (1.8)\\ m\phi^{(iv)}(1) + \gamma\phi''(1) + \phi'''(1) &= -\epsilon\psi''(1), && (1.9)\\ m\psi''(1) + \gamma\psi(1) + \psi'(1) &= -\epsilon\phi(1). && (1.10) \end{aligned}$$

where m and γ are positive constants, and where ϵ is a small parameter with $0 < \epsilon \ll 1$. The functions ϕ and ψ represent the initial displacement of the string and the initial velocity of the string respectively.

The main goal of this paper is to construct explicit, asymptotic approximations for the solution of the problem (1.1) - (1.5) up to order ϵ on a time-scale of order ϵ^{-1} . This paper is organized as follows. In section 2 the proof of the well-posedness of the problem, using a semigroup approach, will be outlined. In section 3 a formal approximation of the solution of (1.1) - (1.5) is constructed using a multiple time-scales perturbation method. The asymptotic validity of this formal approximation will be proved in section 4 on a time-scale of order ϵ^{-1}. Finally in section 5 some remarks will be made and some conclusions will be drawn.

2. Well - Posedness of the Problem

To prove the well - posedness of the initial - boundary value problem (1.1) - (1.5) a semigroup approach can be used. Using such a semigroup approach (see also [6,7]) the problem is reformulated in a spatial space $\mathcal{H}$. The boundary conditions and the pde are absorbed in the space. The space together with an inner product forms a Hilbert space. For this problem the inner product has been defined based on the energy of the string, and the Hilbert space $\mathcal{H}$ is called an energy space.

In this Hilbert space we introduce the following auxiliary functions: $a(t) = u(\bullet,\, t)$, $b(t) = u_t(\bullet,\, t)$and $\eta(t) = mu_t(1,t)$, and the unbounded operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ defined by

$$\begin{aligned} \dot{y} &= Ay, && (2.1)\\ y(0) &= \Phi. && (2.2) \end{aligned}$$

where $y = (a,b,\eta) \in D(A)$.

We have shown in [4] that if $\phi(x) \in H^3(0,1)$, $\phi(0) = \phi''(0) = 0$ and $\psi(x) \in H^2(0,1) \cap V$ then the problem (1.1) - (1.5) has a unique and twice continuously differentiable solution for $x \in [0,1]$ and $t \geq 0$, where $V := \left\{a \in H^1[0,1],\, a(0) = 0\right\}$. Moreover, this solution depends continuously on the initial values. A complete proof can be found in [3].

3. An Approximation of The Solution of The Problem

In this section, an approximation of (1.1) with boundary conditions (1.2) - (1.3) and initial conditions (1.4) - (1.5) will be constructed using a two-time-scales perturbation method. If we expand the solution in a Taylor series with respect to ϵ

$$u(x,t) = u_o(x,t) + \epsilon u_1(x,t) + \epsilon^2 u_2(x,t) + \epsilon^3 \cdots, \tag{3.1}$$

the approximation of the solution of the problem will contain secular terms. However, it can be proved by using the energy of the string that the solution of the problem is bounded. So, secular terms should be avoided. That is why a two-time-scales perturbation method (see also [1,8]) will be used. In using a two - time-scales perturbation method the function $u(x,t)$ is supposed to be a function of x, t and $\tau = \epsilon t$. For that reason, we put

$$u(x,t) = \hat{u}(x,t,\tau;\epsilon). \tag{3.2}$$

Since ϵ is a small parameter it is reasonable to assume that $\hat{u}$, ϕ, and ψ have an infinite series representation of the form

$$\hat{u}(x,t,\tau;\epsilon) \approx u_o(x,t,\tau) + \epsilon u_1(x,t,\tau) + \epsilon^2 u_2(x,t,\tau) + \epsilon^3 \cdots. \tag{3.3}$$

$$\phi(x) \approx \phi_o(x) + \epsilon\phi_1(x) \cdots, \tag{3.4}$$

$$\psi(x) \approx \psi_o(x) + \epsilon\psi_1(x) \cdots. \tag{3.5}$$

Using (3.2) and removing hats and equating the coefficients of the like powers in ϵ then it follows that the solution for u_o is given by

$$u_o(x,t,\tau) = \sum_{n=1}^{\infty} \left(A_n(\tau)\sin(\sqrt{\lambda_n}t) + B_n(\tau)\cos(\sqrt{\lambda_n}t) \right) \sin(\sqrt{\lambda_n}x), \tag{3.6}$$

where λ_n satisfies the relation

$$\cot(\sqrt{\lambda}) = \frac{m\lambda - \gamma}{\sqrt{\lambda}}. \tag{3.7}$$

The problem for u_1 should satisfy

$$u_{1_{tt}} - u_{1_{xx}} = -2u_{o_{t\tau}},\ 0 < x < 1,\ t > 0, \tag{3.8}$$

$$u_1(0,t,\tau) = 0, \quad t > 0,\ \tau > 0,$$

$$mu_{1_{tt}} + \gamma u_1 + u_{1_x} = -2mu_{o_{t\tau}} - u_{o_t}, \quad x = 1,\ t > 0,\ \tau > 0, \tag{3.9}$$

$$u_1(x,0,0) = \phi_1(x), \quad 0 < x < 1, \tag{3.10}$$

$$u_{1_t}(x,0,0) = \psi_1(x) - u_{o_\tau}(x,0,0), \quad 0 < x < 1. \tag{3.11}$$

To solve (3.8) - (3.11) the eigenfunction expansion approach will be used. Making boundary conditions homogeneous is the classical way to solve a wave (or a beam) equation using an eigenfunction expansion. For the non-classical boundary condition at $x = 1$ this classical approach of making the boundary condition homogeneous can not be applied when we apply the eigenfunction expansion to solve the initial-boundary value problem (3.8) - (3.11) the left-hand side of (3.8) for $x = 1$ and that of (3.9) are of the same form. So, to solve the problem correctly the right-hand sides of (3.8) for $x = 1$ and that of (3.9) should be proportional. For that reason we introduce the following transformation

$$u_1(x,t,\tau) = xg(t,\tau) + v(x,t,\tau), \tag{3.12}$$

and then we expand v into the eigenfunction expansion

$$v(x,t,\tau) = \sum_{n=1}^{\infty} v_n(t,\tau)\sin(\sqrt{\lambda_n}x). \tag{3.13}$$

After some calculations it follows that

$$g(t,\tau) = -\frac{1}{\gamma+1}u_{o_t}(1,t,\tau). \tag{3.14}$$

Using the function g given by (3.14) the function v defined by (3.12) satisfies both the partial differential equation (3.8) and the boundary condition (3.9) automatically.

Remark 3.1. If $m = \gamma = 0$ then the transformation (3.12) and the classical way of making boundary conditions homogeneous coincide.

After expanding x into a Fourier series we obtain the differential equations for v_n, namely

$$\begin{aligned} v_{n_{tt}} + \lambda_n v_n &= (2\sqrt{\lambda_n}A'_n + \frac{c_n}{\gamma+1}\lambda_n^{3/2}\sin(\sqrt{\lambda_n})A_n)\sin(\sqrt{\lambda_n}t) \\ &\quad -(2\sqrt{\lambda_n}B'_n + \frac{c_n}{\gamma+1}\lambda_n^{3/2}\sin(\sqrt{\lambda_n})B_n)\cos(\sqrt{\lambda_n}t) \\ &\quad +\frac{c_n}{\gamma+1}\sum_{\substack{p=1\\ \neq n}}^{\infty}\lambda_p^{3/2}\sin(\sqrt{\lambda_p})(A_p\sin(\sqrt{\lambda_p}t) - B_p\cos(\sqrt{\lambda_p}t)), \end{aligned} \tag{3.15}$$

where $c_n = \frac{2(\gamma+1)\sin(\sqrt{\lambda_n})}{\lambda_n+(m\lambda_n+\gamma)\sin^2(\sqrt{\lambda_n})}$. In order to remove secular terms, it follows from (3.15) that A_n and B_n have to satisfy

$$\begin{aligned} A'_n + \frac{c_n}{2(\gamma+1)}\lambda_n\sin(\sqrt{\lambda_n})A_n &= 0, \\ B'_n + \frac{c_n}{2(\gamma+1)}\lambda_n\sin(\sqrt{\lambda_n})B_n &= 0, \end{aligned} \tag{3.16}$$

now define

$$\alpha_n = \frac{\lambda_n\sin^2(\sqrt{\lambda_n})}{\lambda_n + (m\lambda_n+\gamma)\sin^2(\sqrt{\lambda_n})} > 0. \tag{3.17}$$

The solution of (3.16) is given by

$$A_n(\tau) = A_n(0)\exp(-\alpha_n\tau), \tag{3.18}$$

$$B_n(\tau) = B_n(0)\exp(-\alpha_n\tau). \tag{3.19}$$

From (1.6) - (1.10) and (3.17) - (3.19) it follows that the infinite series representation for u_o is twice continuously differentiable with respect to x and t, and infinitely many times with respect to τ.

Remark 3.2. We know that $\cot(\lambda_n)$ tends to ∞ as n tends to ∞. So, it means that α_n tends to 0 as n tends to ∞. So, from (3.18) and (3.19) it follows that the solution u_o is stable but not uniform.

Using (3.15) - (3.16) and (3.12) u_1 can be determined, yielding

$$u_1(x,t,\tau) = \sum_{n=1}^{\infty}\left(v_n(t,\tau)+\frac{c_n}{\gamma+1}\sum_{p=1}^{\infty}\sqrt{\lambda_p}\sin(\sqrt{\lambda_p})(A_p(\tau)\sin(\sqrt{\lambda_p}t)-B_p(\tau)\cos(\sqrt{\lambda_p}t)\right)\sin(\sqrt{\lambda_n}x).$$

It should be remarked that u_1 still contains infinitely many undetermined functions D_n and E_n of τ in $v_n(t,\tau)$, $n = 1,2,3,\cdots$. These functions have to be used to avoid secular terms in u_2. However, it is our goal to construct a function $\bar{u}$ that satisfies the differential equation, and boundary and initial conditions up to order ϵ^2. For that reason, D_n and E_n are taken to be equal to their initial values.

Again from (1.6) - (1.10) it follows that the infinite series representation for u_1 is twice continuously differentiable with respect to x and t and infinitely many times with respect to τ.

4. On The Asymptotic Validity of Formal Approximations

In this section, the asymptotic validity of the formal approximation on the interval $0 \leq t \leq L_o\epsilon^{-1}$ and $0 \leq x \leq 1$ is studied briefly. We have constructed the function

$$\bar{u}(x,t;\epsilon) = u_o(x,t,\tau) + \epsilon u_1(x,t,\tau). \tag{4.1}$$

This function is a so-called formal approximation of the solution and this function satisfies the following initial-boundary value problem

$$
\begin{aligned}
(4.2)\qquad & \bar{u}_{tt} - \bar{u}_{xx} &=& \ \epsilon^2 F(x,t;\epsilon),\ \ 0 < x < 1,\ t > 0,\\
& \bar{u}(0,t) &=& \ 0, \qquad t \geq 0,\\
(4.3)\qquad & m\bar{u}_{tt} + \gamma\bar{u} + \bar{u}_x + \epsilon\bar{u}_t &=& \ \epsilon^2 R(x,t;\epsilon), \quad x = 1,\ t \geq 0,\\
& \bar{u}(x,0) &=& \ \phi_o(x) + \epsilon\phi_1(x), \quad 0 < x < 1,\\
(4.4)\qquad & \bar{u}_t(x,0) &=& \ \psi_o(x) + \epsilon\psi_1(x) + \epsilon^2 u_{1_\tau}(x,0,0),\ 0 < x < 1,
\end{aligned}
$$

By doing the same as in section 2 we can derive an integral representation for $\bar{u}$. For $0 \leq t \leq L_o\epsilon^{-1}$ and $0 \leq x \leq 1$, we can estimate the difference between u and $\bar{u}$ by subtracting the integral representations for u and $\bar{u}$. It can also be shown that the difference is of order ϵ on a time-scale of order ϵ^{-1}. For detailed calculations and proofs we refers to [3].

5. Conclusions

In this paper, an initial - boundary value problem for a weakly damped string has been considered. It can be shown that, using a semigroup approach, the initial -boundary value problem (1.1) - (1.5) is well-posed for $0 \leq x \leq 1$ and $t \geq 0$. Using the energy integral we can prove that the solution is bounded. The construction of the approximation is far from being an easy task. It is not possible to solve (3.8) - (3.11) in the classical way by making the boundary condition at $x = 1$ homogeneous. This is due to the non-classical boundary condition at $x = 1$. It can only be done by balancing the right-hand side of (3.8) and that of (3.9) by transforming u in an appropriate way (3.12) . It also should be noted that the way to solve the wave equation with non-classical boundary condition using the eigenfunction expansion as we have done in section 3 is an extension of the classical way (*remark* 3.1). Finally, we proved that the formal approximation is an asymptotic one on a time-scale of order ϵ^{-1}. Although we did not consider external forces (leading to an inhomogeneous PDE) these problems can be solved in a similar way using the balancing procedure as given in section 3. Actually by considering (3.8) - (3.11) we have solved an inhomogeneous problem. The results presented in this paper may be extended to weakly nonlinear pde, and to other non-classical boundary conditions.

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